Toric varieties associated with weighted graphs

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Abstract. We give an algorithm to associate a toric variety with a given weighted graph whose incidence matrix is negative definite.

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1 Introduction

By plumbing, a weighted graph can be associated with a configuration of curves embedded in a smooth surface (see [6], p.120). If the incidence matrix of the weighted graph is negative definite, the contraction of the corresponding configuration of curves gives a singularity of a normal analytic surface (see [4]). Conversely, the incidence matrix of the dual graph associated with the exceptional divisor of a resolution of a normal (analytic or algebraic) surface is negative definite (see [2, 12]).

Let Γ be a weighted graph whose the incidence matrix is negative definite. In [9], Lipman studied a certain set of positive divisors supported on Γ. We will call this set the *semigroup of Lipman*. The relation between the elements of the semigroup of Lipman and the functions in the maximal ideal of the singularity corresponding to Γ (see [9, 1, 14]) guides us to look for more information on the structure of that semigroup. By this aim, we follow a suggestion of Pinkham which appears in [11] (see remarque in p.166). This leads us to associate a toric variety with Γ by using the semigroup of Lipman of Γ. The toric variety obtained in this way is affine and has only quotient singularities.

In the first part of this work, we introduce the semigroup of Lipman and its "generators" and, in the second part, we describe in detail the construction of the toric varieties associated with the weighted graphs corresponding to the rational double points of a surface.

2 Weighted graphs and semigroup of Lipman

Let Γ be a graph without loops, with vertices $C_1, \cdots, C_n$, weighted by pairs (w_i, g_i) at each vertex C_i, $(1 \leq i \leq n)$, where w_i is a positive integer called the weight of C_i, and g_i is a non-negative integer, called the genus of C_i.

We associate a symmetric matrix $\mathcal{M}(\Gamma) = (\alpha_{ij})_{1\leq i,j\leq n}$, with Γ in the following way: $\alpha_{ii} = -w_i$ and α_{ij} is the number of edges linking the vertices C_i and C_j whenever $i \neq j$. We call $\mathcal{M}(\Gamma)$ the *incidence matrix* of Γ. A weighted graph is called a *singular graph* if the associated incidence matrix is negative definite.

By plumbing, a weighted graph defines a (non-unique) complex curve configuration embedded in a nonsingular complex analytic surface, such that each vertex C_i is represented by a curve of genus g_i and of the self-intersection $-w_i$. This correspondance gives the following relation between singular graphs and singularities of surfaces:

Theorem 2.1 (see [4], p.367) *If Γ is a singular graph, then there is a normal complex analytic surface singularity and a resolution of this singularity such that its exceptional divisor is the configuration of curves corresponding to Γ.*

Conversely, we have:

Theorem 2.2 (see [2, 12]) *The incidence matrix of the dual graph associated with the exceptional divisor of a resolution of a complex normal surface singularity is negative definite.*

Now, let $\mathcal{G}$ denotes the free abelian group generated by the vertices C_i of Γ, that is, the set of divisors with integers coefficients:

$$\mathcal{G} = \{\sum_{i=1}^{n} m_i C_i \mid m_i \in \mathbb{Z}\}.$$

When $m_i \in \mathbb{N}$ the elements of $\mathcal{G}$ are called the *positive divisors* supported on Γ. As in [9], consider the set

$$\mathcal{E}^+(\Gamma) = \{Y \in \mathcal{G} \mid (Y \cdot C_i) \leq 0 \quad \text{for all } i\}.$$

Zariski proved that the subset $\mathcal{E}^+(\Gamma)$ of $\mathcal{G}$ is not empty if Γ is a singular graph (see [15]). Notice that the inequality $(Y \cdot C_i) \leq 0$ implies $m_i \geq 1$ for all i and for any element $Y = \sum m_i C_i$ in $\mathcal{E}^+(\Gamma)$. Moreover, we can easily show that:

Proposition-Definition 2.3 *The set $\mathcal{E}^+(\Gamma)$ is a semigroup and we call it the semigroup of Lipman.*

A partial order on $\mathcal{E}^+(\Gamma)$ is defined as follows: Given two elements $Y_1 = \sum_{i=1}^n a_i C_i$ and $Y_2 = \sum_{i=1}^n b_i C_i$ of $\mathcal{E}^+(\Gamma)$, we say $Y_1 \leq Y_2$ if $a_i \leq b_i$ for all i. In [7], (see proposition 4.1), Laufer gives an algorithm to calculate the smallest element of $\mathcal{E}^+(\Gamma)$. The same algorithm permits us to find all other elements in $\mathcal{E}^+(\Gamma)$ (see [11, 14]).

Let Γ be a singular graph. Hence $\mathcal{E}^+(\Gamma) \neq \emptyset$. We want to look for the generators of $\mathcal{E}^+(\Gamma)$: Let $Y' = \sum_{i=1}^n m_i C_i$ be any positive divisor in $\mathcal{G}$. Consider

$$\mathcal{M}(\Gamma) \cdot (m_1, m_2, \ldots, m_n)^t = (y_1, y_2, \ldots, y_n)^t.$$

This says that $(Y' \cdot C_i) = y_i$. Let us denote by δ_i the column matrix with coefficients 0 everywhere except in the i-th row, where the entry is -1, and consider $\mathcal{M}(\Gamma) \cdot (m_{i1}, m_{i2}, \cdots, m_{in})^t = \delta_i$. Put $F_i' = \sum_{j=1}^n m_{ij} C_j$. We have $m_{ij} \in \mathbb{Q}^+$. We can write F_i' as $F_i' = k_i \cdot F_i$ such that k_i denotes the least common factor of the denominators of the coefficients m_{ij} for $j = 1, \cdots, n$. Then we can conclude:

Proposition 2.4 *With the preceding notation, the divisor F_i is an element of $\mathcal{E}^+(\Gamma)$.*

Moreover, we have:

Theorem 2.5 *There exists a subset $G^+(\Gamma) = \{F_1, \ldots, F_n\}$ of $\mathcal{E}^+(\Gamma)$ such that each element in $\mathcal{E}^+(\Gamma)$ can be written as a linear combination of the elements of $G^+(\Gamma)$ with coefficients in $\mathbb{Q}^+$.*

Proof. Let G be the semigroup generated by the elements of $G^+(\Gamma)$ with coefficients in $\mathbb{Q}^+$ and $\mathcal{G}$ as defined above. To prove that $\mathcal{E}^+(\Gamma) = \mathcal{G} \cap G$, first we show that an element $Y = \sum_{i=1}^n a_i C_i$ in $\mathcal{E}^+(\Gamma)$ is contained in $\mathcal{G} \cap G$. For that, we consider $\mathcal{M}(\Gamma) \cdot Y = \sum_{i=1}^n y_i$, where $(Y \cdot C_i) = y_i$. We have $y_i \leq 0$ for all i. Now take a divisor $D = \sum_{i=1}^n d_i C_i$ in G. So $d_i \in \mathbb{Q}^+$ for all i. Assume that $\mathcal{M}(\Gamma) \cdot D = \delta_i$. Since a negative definite matrix is invertible, we obtain $Y = -\sum_{i=1}^n d_i y_i C_i$. Since $y_i \leq 0$ for all i, the coefficient $-d_i y_i$ for all i is in $\mathbb{Q}^+$. Then, we have $Y \in \mathcal{G} \cap G$. Second, we show the inclusion $\mathcal{G} \cap G \subset \mathcal{E}^+(\Gamma)$. For this, we start with $D \in \mathcal{G} \cap G$. This means that $D = \sum_{i=1}^n b_j F_j$ with $b_j \in \mathbb{Q}^+$. Consider $(D \cdot C_i) = \sum_{i=1}^n b_j (F_j \cdot C_i)$. By the hypothesis, each element F_j in $G^+(\Gamma)$ satisfies the condition $(F_j \cdot C_i) \leq 0$ for all i. Since $b_j \in \mathbb{Q}^+$ for all j, we have $(D \cdot C_i) \leq 0$ for all i, hence $D \in \mathcal{E}^+(\Gamma)$. Then, we have the equality $\mathcal{E}^+(\Gamma) = \mathcal{G} \cap G$. □

3 Toric varieties associated with $\mathcal{E}^+(\Gamma)$

Let Γ be a singular graph with vertices $C_1, \cdots, C_n$. In this section, we will construct the toric variety corresponding to Γ by using $\mathcal{E}^+(\Gamma)$. We will describe our algorithm on some examples.

Let $N =< C_1, \ldots, C_n >_{\mathbb{Z}}$ be a lattice generated by the vertices of Γ and $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice generated by $C_1^*, C_2^*, \ldots, C_n^*$ such that $(C_i, C_j^*) = 1$ if $i = j$, 0 otherwise. We denote $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $\mathbb{N}_{\mathbb{R}}$ and the dual space $M \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$. Let σ in $\mathbb{N}_{\mathbb{R}}$ be the rational polyhedral cone defined by $\mathcal{E}^+(\Gamma)$ and $\check{\sigma}$ in $M_{\mathbb{R}}$ be the dual cone of σ. The semigroup $\check{\sigma} \cap M$ is the set $\{u \in M \mid (u, v) \geq 0 \quad \text{for all } v \in \sigma\}$; it is finitely generated. The corresponding variety $\mathrm{Spec}\, \mathbb{C}[\check{\sigma} \cap M]$ is an affine toric variety (see [3] for details). We will denote

$\mathbb{C}[\check{\sigma} \cap M]$ by $\mathbb{C}[x_1, \ldots, x_n]$. Here we will give an algorithm to find out explicitly $\mathbb{C}[\check{\sigma} \cap M]$ for a given Γ.

Denote by $F_1, \ldots, F_n$ the generators of $\mathcal{E}^+(\Gamma)$. Let $N' =< F_1, \ldots, F_n >_{\mathbb{Z}}$ be a lattice and $M' = Hom(N', \mathbb{Z})$ be its dual lattice generated by $F_1^*, F_2^*, \ldots, F_n^*$ such that $(F_i, F_j^*) = 1$ if $i = j$, 0 otherwise. We obtain:

Proposition 3.1 *The dual lattice M' of N' is generated by the rows of $\mathcal{M}(\Gamma)$ multiplied by $\hat{k}_i = -k_i/det\mathcal{M}(\Gamma)$, where ${F_i}' = -\hat{k}_i F_i$.*

The proposition follows from the construction of the F_i'. In other words, let $F_i' = \sum_{j=1}^n a_{ij} C_j$ such that $\mathcal{M}(\Gamma) \cdot (a_{i1}, \cdots, a_{in}) = \delta_i$. Since $M(\Gamma)$ is invertible, we can write $(a_{i1}, \cdots, a_{in}) = \frac{1}{\det \mathcal{M}(\Gamma)}(b_{i1}, \ldots, b_{in})$. By taking the greatest common divisors k_i of the b_{ij} for $j = 1, \ldots, n$, we get $F_i' = \frac{k_i}{\det \mathcal{M}(\Gamma)} F_i$ where $F_i = \sum_{j=1}^n b'_{ij} C_j$ such that $k_i b'_{ij} = b_{ij}$. Such the F_i's are the generators of $\mathcal{E}^+(\Gamma)$.

The lattice N' is a sublattice of finite index of N and $\check{\sigma} \cap M \subset \check{\sigma} \cap M'$. Then, we have the following proposition:

Proposition 3.2 *With the preceding notation,* $\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[M']^{\frac{N}{N'}}$.

Proof. See Oda [10, corollary 1.16] or page 34 in Fulton [3]. □

As a result of proposition 3.2 we have:

Corollary 3.3 *The affine variety* $\operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ *has only quotient singularities.*

To find out explicitly the algebra $\mathbb{C}[\check{\sigma} \cap M]$ corresponding to Γ, first of all, by using proposition 3.1, we determine the algebra $\mathbb{C}[M']$ and then, the ring of invariants $\mathbb{C}[M']^{\frac{N}{N'}}$ under the N/N'-action (see proposition 3.2). In what follows, we will do this construction in detail for the weighted graphs A_n, D_n, E_6, E_7 and E_8 (or briefly ADE trees).

3.1 Construction for ADE Trees

It is well known that ADE trees are singular trees and the configuration of curves associated with an ADE tree is the exceptional divisor of a resolution of a rational double point of a surface.

I. Let Γ be a weighted graph of type A_{n-1} for $n \geq 2$. The incidence matrix $\mathcal{M}(A_{n-1})$ is defined by $(C_i \cdot C_j) = 1$ if $\mid j-i \mid= 1$, -2 if $i=j$ and 0 otherwise. With the notation of the preceding section, we obtain

$$F_i' = \frac{1}{n}[(n-i)C_1 + 2(n-i)C_2 + \ldots + i(n-i)C_i + i(n-i-1)C_{i+1} + \ldots + iC_{n-1}]$$

This gives $F_i' = \frac{d_i}{n} F_i$ with $d_i = (n-i,i)$. If n is odd, then $\det \mathcal{M}(A_{n-1}) = n, k_1 = k_{n-1} = 1$ and $k_i = d_i$; and if n is even, then $\det \mathcal{M}(A_{n-1}) = -n, k_1 = k_{n-1} = -1$ and $k_i = -d_i$; in both cases, $\hat{k}_i = -\frac{d_i}{n}$. Note that $k_i = k_{n-i}$ for all i. If we denote $\mathbb{C}[M'] := \mathbb{C}[u_1, \ldots, u_{n-1}]$, we have

$$u_1 = x_1^{-2\hat{k}_1} x_2^{\hat{k}_1}, u_2 = x_1^{\hat{k}_2} x_2^{-\hat{k}_2} x_3^{\hat{k}_2}, \ldots,\ u_i = x_{i-1}^{\hat{k}_i} x_i^{-2\hat{k}_i} x_{i+1}^{\hat{k}_i}, \ldots,$$
$$u_{n-2} = x_{n-3}^{\hat{k}_2} x_{n-2}^{-2\hat{k}_2} x_{n-1}^{\hat{k}_2}, u_{n-1} = x_{n-2}^{\hat{k}_1} x_{n-1}^{-2\hat{k}_1}.$$

Recall that $\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[x_1, \ldots, x_{n-1}]$. On the other hand, the finite group N/N' is described as follows:

Proposition 3.4 *The group N/N' is generated by $\overline{C}_1, \ldots, \overline{C}_{n-1}$ over $\mathbb{Z}$ such that the order of $\overline{C}_i$ equals $\frac{n}{(d_{i-1}, 2d_i, d_{i+1})}$ for $i = 2, \ldots, n-2$ and the order of $\overline{C}_1$ and $\overline{C}_{n-1}$ equal $\frac{n}{(2,d_2)}$, where $\overline{C}_i = C_i + N'$.*

Proof. To calculate the order of the $\overline{C}_i$ in N/N', denote by r_i, it suffices to show that some multiples of the C_i are in N': Let $\overline{F} \in N/N'$, where $\overline{F} = F + N'$. Suppose that $F \in N'$. So, there exists $a_i \in \mathbb{Z}$ such that $F = a_1F_1 + a_2F_2 + \cdots + a_{n-1}F_{n-1}$. Now, we rewrite F in terms of the C_i by replacing the F_i. Thus we obtain $F = b_1C_1 + \cdots + b_{n-1}C_{n-1}$ where

$$b_1 = (a_1(n-1) + \frac{a_2}{d_2}(n-2) + \frac{a_3}{d_3}(n-3) + \cdots + \frac{a_{n-3}}{d_3}3 + \frac{a_{n-2}}{d_2}2 + a_{n-1}),$$
$$b_2 = (a_1(n-2) + \frac{a_2}{d_2}(n-2)2 + \frac{a_3}{d_3}2(n-3) + \cdots +$$
$$\frac{a_{n-3}}{d_3}2 \cdot 3 + \frac{a_{n-2}}{d_2}2 \cdot 2 + a_{n-1}2), \cdots ,$$
$$b_{n-1} = (a_1 + \frac{a_2}{d_2}2 + \frac{a_3}{d_3}3 \cdot 1 + \cdots + \frac{a_{n-3}}{d_3}(n-3) +$$
$$\frac{a_{n-2}}{d_2}(n-2) + a_{n-1}(n-1)).$$

Then, by solving the a_i in terms of the b_i, we get

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 2b_1 - b_2 \\ -d_2b_1 + 2d_2b_2 - d_2b_3 \\ \vdots \\ -d_ib_{i-1} + 2d_ib_i - d_ib_{i+1} \\ \vdots \\ -d_2b_{n-3} + 2d_2b_{n-2} - d_2b_{n-1} \\ -b_{n-2} + 2b_{n-1} \end{pmatrix}, \tag{3.1}$$

where $a_i \in \mathbb{Z}$. If we assume that $b_1 \neq 0$ and $b_2 = \cdots = b_{n-1} = 0$, then we obtain $a_1F_1 + \cdots + a_{n-1}F_{n-1} = b_1C_1$. Hence the equality (3.1) gives

$$a_1 = \frac{2b_1}{n}, \quad a_2 = \frac{-d_2b_1}{n} \quad \text{and} \quad a_3 = \cdots = a_{n-1} = 0.$$

Thus, the smallest integer b_1 such that $a_1, a_2 \in \mathbb{Z}$ is the order r_1 of $\overline{C}_1$ in N/N'. Now, it is easy to observe that $r_1 = \frac{n}{(2,d_2)}$. If we repeat this process for $b_i \neq 0$ and $b_1 = \cdots = b_{i-1} = b_{i+1} = \cdots = b_{n-1} = 0$ ($i = 2, \ldots, n-2$), then the equality (3.1) will give:

$$a_{i-1} = \frac{-d_{i-1}b_i}{n}, \quad a_i = \frac{-2d_ib_i}{n}, \quad a_{i+1} = \frac{-d_{i+1}b_i}{n},$$
$$a_1 = a_2 = \cdots = a_{i-3} = a_{i-2} = a_{i+2} = a_{i+3} = \cdots = a_{n-1} = 0.$$

Hence the smallest b_i such that $a_{i-1}, a_i, a_{i+1} \in \mathbb{Z}$ is the order r_i of $\overline{C}_i$ in N/N'. It can easily be observed that $r_i = \frac{n}{(d_{i-1},d_i,d_{i+1})}$. When $i = n-1$, we obtain

$$a_{n-2} = \frac{-d_2b_{n-1}}{n}, \quad a_{n-1} = \frac{2b_{n-1}}{n}, \quad a_1 = a_2 = \cdots = a_{n-3} = 0.$$

Hence $r_{n-1} = \frac{n}{(2,d_2)}$. $\square$

Corollary 3.5 *For $i = 1, \ldots, n-1$, we have $r_i = r_{n-i}$ and $r_i = n$ or $\frac{n}{2}$.*

Proof. Since $d_i = d_{n-i}$ we have $r_i = r_{n-i}$. Since $(d_{i-1}, 2d_i, d_{i+1})$ equals either 1 or 2, we have $r_i = n$ or $\frac{n}{2}$ respectively. $\square$

Theorem 3.6 *If σ is the cone defined by the semigroup of Lipman $\mathcal{E}^+(A_{n-1})$, $(n \geq 2)$, then the algebra $\mathbb{C}[\check{\sigma} \cap M]$ is generated by*

$$u_i^{\frac{n}{d_i}}, u_1^i u_2^{\frac{\overline{2i}}{d_2}} u_3^{\frac{\overline{3i}}{d_3}} \cdots u_{n-i}^{\frac{\overline{(n-i)i}}{d_{n-i}}} u_{n-i+1}^{\frac{\overline{(n-i)(i-1)}}{d_{n-i+1}}} \cdots u_{n-2}^{\frac{\overline{(n-i)2}}{d_2}} u_{n-1}^{(n-i)}$$

for all $i = 1, \ldots, n-1$, where $-$ denotes modulo n.

Proof. Let us denote $\overline{C}_i$ by η_i for each i. Thus $\eta_i = (\exp 2\pi i)^{\frac{1}{r_i}}$ such that $\eta_i^{r_i} = 1$. Now, the N/N'-action on the coordinates of $\mathbb{C}[M']$ (see p. 34 in [3]) is the following:

$$\begin{aligned}
\eta_1(u_1, \ldots, u_{n-1}) &= (\eta_1^{\frac{2}{(d_2,2)}} u_1, \eta_1^{\frac{-d_2}{(d_2,2)}} u_2, u_3, \ldots, u_{n-1}), \\
\eta_2(u_1, \ldots, u_{n-1}) &= (\eta_2^{\frac{-1}{(1,2d_2,d_3)}} u_1, \eta_2^{\frac{2d_2}{(1,2d_2,d_3)}} u_2, \eta_2^{\frac{-d_3}{(1,2d_2,d_3)}} u_3, u_4, \ldots, u_{n-1}), \\
&\ \ \vdots \\
\eta_i(u_1, \ldots, u_{n-1}) &= (u_1, \ldots, u_{i-2}, \eta_i^{\frac{-d_{i-1}}{(d_{i-1},2d_i,d_{i+1})}} u_{i-1}, \eta_i^{\frac{2d_i}{(d_{i-1},2d_i,d_{i+1})}} u_i, \\
&\quad \eta_i^{\frac{-d_{i+1}}{(d_{i-1},2d_i,d_{i+1})}} u_{i+1}, u_{i+2}, \ldots, u_{n-1}), \\
&\ \ \vdots \\
\eta_{n-2}(u_1, \ldots, u_{n-1}) &= (u_1, \ldots, u_{n-4}, \eta_{n-2}^{\frac{-d_3}{(d_3,2d_2,1)}} u_{n-3}, \eta_{n-2}^{\frac{2d_2}{(d_3,2d_2,1)}} u_{n-2}, \\
&\quad \eta_{n-2}^{\frac{-1}{(d_3,2d_2,1)}} u_{n-1}), \\
\eta_{n-1}(u_1, \ldots, u_{n-1}) &= (u_1, \ldots, u_{n-3}, \eta_{n-1}^{\frac{-d_2}{(d_2,2)}} u_{n-2}, \eta_{n-1}^{\frac{2}{(d_2,2)}} u_{n-1}).
\end{aligned}$$

For $u = u_1^{a_1} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} \in \mathbb{C}[M']$, the action above gives:

$$
\begin{aligned}
\eta_1(u) &= \eta_1^{\frac{2a_1-d_2a_2}{(d_2,2)}}u,\\
\eta_2(u) &= \eta_2^{\frac{-a_1+2d_2a_2-d_3a_3}{(1,2d_2,d_3)}}u,\\
&\vdots\\
\eta_i(u) &= \eta_i^{\frac{-d_{i-1}a_{i-1}+2d_ia_i-d_{i+1}a_{i+1}}{(d_{i-1},2d_i,d_{i+1})}}u,\\
&\vdots\\
\eta_{n-2}(u) &= \eta_{n-2}^{\frac{-d_3a_{n-3}+2d_2a_{n-2}-a_{n-1}}{(1,2d_2,d_3)}}u,\\
\eta_{n-1}(u) &= \eta_{n-1}^{\frac{-d_2a_{n-2}+2a_{n-1}}{(d_2,2)}}u.
\end{aligned}
$$

Since the ring of invariants is determined by the smallest $a_1, \dots, a_{n-1}$ satisfying $\eta_i(u) = u$ for all $i = 1, \dots, n-1$, then the action of the group on the product gives the following system of equations:

$$
\begin{aligned}
\frac{2a_1 - d_2a_2}{(d_2,2)} &= r_1l_1,\\
\frac{-a_1 + 2d_2a_2 - d_3a_3}{(1,2d_2,d_3)} &= r_2l_2,\\
&\vdots\\
\frac{-d_{i-1}a_{i-1} + 2d_ia_i - d_{i+1}a_{i+1}}{(d_{i-1},2d_i,d_{i+1})} &= r_il_i,\\
&\vdots\\
\frac{-d_3a_{n-3} + 2d_2a_{n-2} - a_{n-1}}{(1,2d_2,d_3)} &= r_{n-2}l_{n-2},\\
\frac{-d_2a_{n-2} + 2a_{n-1}}{(d_2,2)} &= r_{n-1}l_{n-1}
\end{aligned}
$$

for some l_i. Recall that $r_i = \operatorname{ord}\eta_i$. If we choose $u = u_i^{a_i}$, then the conditions $\eta_j(u) = u$ for all j implies that a_i must be $0, 1, \dots, \frac{n}{d_i} - 1$ modulo $\frac{n}{d_i}$. To find $a_1, \dots, a_{n-1}$, we first consider $a_1 = 1$ and, we substitute this value of a_1 into the system above. Thus we get a_2. Then, from a_2 we find a_3 and so on. Repeating the same process for different values of a_1, we can have all possible solutions $(a_1, a_2, \dots, a_{n-1}, a_n)$ of the system of equations above. This gives the theorem. □

II. Let Γ be a weighted graph of type D_n, $(n \geq 4)$, with vertices $C_1, \ldots, C_n$. The incidence matrix $\mathcal{M}(D_n)$ is defined by $(C_i \cdot C_j) = 1$ if $j = i + 1$ for $i = 1, \ldots, n-2, (C_{n-2} \cdot C_n) = 1$ and 0 otherwise. We have $\det \mathcal{M}(D_n) = 4$ when n is even and, $\det \mathcal{M}(D_n) = -4$ when n is odd. Hence

$$\begin{aligned} F'_i =& C_1 + 2C_2 + \cdots + (i-1)C_{i-1} + i(C_i + C_{i+1} + \cdots + C_{n-2}) \\ &+ \frac{i}{2}(C_{n-1} + C_n) \text{ for } i \leq n-2, \\ F'_{n-1} =& \frac{1}{2}(C_1 + 2C_2 + \cdots + (n-2)E_{n-2}) + \frac{n}{4}C_{n-1} + \frac{(n-2)}{4}C_n \quad \text{and} \\ F'_n =& \frac{1}{2}(C_1 + 2C_2 + \cdots + (n-2)C_{n-2}) + \frac{(n-2)}{4}C_{n-1} + \frac{n}{4}C_n. \end{aligned}$$

Thus $\mathcal{E}^+(D_n)$ is generated by $F_1, \cdots, F_n$ such that $F'_i = \frac{k_i}{-4}F_i$ if n is even, where

$$k_i = \begin{cases} 2 & \text{if } 1 \leq i \leq n-2 \text{ and } i \text{ is odd}, \\ 4 & \text{if } 1 \leq i \leq n-2 \text{ and } i \text{ is even}, \\ 2 & i = n-1, n; \end{cases}$$

and $F'_i = \frac{k_i}{-4}F_i$ if n is odd, where

$$k_i = \begin{cases} -2 & \text{if } 1 \leq i \leq n-2 \text{ and } i \text{ is odd}, \\ -4 & \text{if } 1 \leq i \leq n-2 \text{ and } i \text{ is even}, \\ -1 & i = n-1, n. \end{cases}$$

This gives $\mathbb{C}[M'] = \mathbb{C}[u_1, \ldots, u_n]$ such that

$$u_1 = x_1^{-2\hat{k}_1} x_2^{\hat{k}_1}, \quad u_i = x_{i-1}^{\hat{k}_i} x_i^{-2\hat{k}_i} x_{i+1}^{\hat{k}_i} \text{ for } 2 \leq i \leq n-3,$$
$$u_{n-2} = x_{n-3}^{\hat{k}_{n-2}} x_{n-2}^{-2\hat{k}_{n-2}} x_{n-1}^{\hat{k}_{n-2}} x_n^{\hat{k}_{n-2}}, \quad u_{n-1} = x_{n-2}^{\hat{k}_{n-1}} x_{n-1}^{-2\hat{k}_{n-1}}, \quad u_n = x_{n-2}^{\hat{k}_n} x_n^{-2\hat{k}_n}.$$

The finite group N/N' is given as follows:

Proposition 3.7 *With the preceding notation, we have:*

(i) *If n is even, then N/N' is generated by $\overline{C}_i$ with i even and $1 \leq i \leq n-2$, over $\mathbb{Z}$ such that, for each $1 \leq i \leq n-2$, we have $r_i = 1$ when i is odd, $r_i = 2$ when i is even, and $r_{n-1} = r_n = 1$.*

(ii) *If n is odd, then N/N' is generated by $\overline{C}_i$ with i even and $1 \leq i \leq n-3$, $\overline{C}_{n-2}$, $\overline{C}_{n-1}$ and $\overline{C}_n$, over $\mathbb{Z}$, where, for each $1 \leq i \leq n-3$, $r_i = 1$ when i is odd, $r_i = 2$ when i is even and, $r_{n-2} = 4$ and $r_{n-1} = r_n = 2$.*

Proof. Following the proof of theorem 3.4, we obtain a linear system $(b_1, \cdots, b_n) = (a_1, \cdots, a_n) \cdot \mathcal{A}$ for some invertible matrix $\mathcal{A}$. As in the equality (3.1), we obtain:

$$(a_1, \ldots, a_n)^t = \frac{1}{\det \mathcal{M}(D_n)} \cdot A^{-1} \cdot (b_1, \cdots, b_n)^t \tag{3.2}$$

where $A^{-1} = \mathrm{Diag}(-k_1, -k_2, \cdots, -k_n) \cdot \mathcal{M}(D_n)$. Now, let us assume that $b_1 = b_2 = \cdots = b_{j-1} = b_{j-2} = \cdots = b_n = 0$ and $b_j \neq 0$. Then, we replace the values of k_i and b_i into (3.2) to obtain the $a_i \in \mathbb{Z}$ in terms of b_j. Note that the values of k_i depend on the fact that n is odd or even. We continue as in the case of A_{n-1} and we conclude the proof of the proposition. □

Theorem 3.8 *If σ is the cone defined by the semigroup $\mathcal{E}^+(D_n)$, $(n \geq 4)$, then the algebra $\mathbb{C}[\check{\sigma} \cap M]$ is generated by*

(i) $u_1^2, u_2, u_3^2, u_4, \ldots, u_{n-3}^2, u_{n-2}, u_{n-1}^2, {u_n}^2, u_{n-1}u_n, vu_n, vu_{n-1}$ *if n is even, where $v = u_1 u_3 \cdots u_{n-5} u_{n-3}$;*

(ii) $u_1^2, u_2, u_3^2, u_4, \ldots, u_{n-4}^2, u_{n-3}, u_{n-2}^2, u_{n-1}^4, u_n^4, u_{n-1}^2 u_n^2, vu_{n-1}u_n^3, vu_{n-1}^3 u_n$ *if n is odd, where $v = u_1 u_3 \cdots u_{n-4} u_{n-2}$.*

Proof. The action of the finite group N/N' on the coordinates of $\mathbb{C}[M']$ is given by

$$\begin{aligned}
\eta_i(u_1, \ldots, u_n) &= (u_1, \ldots, u_{i-2}, \eta_i^{r_i \hat{k}_{i-1}} u_{i-1}, \eta_i^{-2r_i \hat{k}_i} u_i, \eta_i^{r_i \hat{k}_{i+1}} u_{i+1}, u_{i+2}, \\
&\qquad \ldots, u_n) \quad \text{ for } i \text{ even and } 2 \leq i \leq n-3, \\
\eta_{n-2}(u_1, \ldots, u_n) &= (u_1, \ldots, u_{n-4}, \eta_{n-2}^{\hat{k}_{n-3} r_{n-2}} u_{n-3}, \eta_{n-2}^{-2\hat{k}_{n-2} r_{n-2}} u_{n-2}, \\
&\qquad \eta_{n-2}^{\hat{k}_{n-1} r_{n-2}} u_{n-1}, \eta_{n-2}^{\hat{k}_n r_{n-2}} u_n), \\
\eta_{n-1}(u_1, \ldots, u_n) &= (u_1, \ldots, u_{n-3}, \eta_{n-1}^{\hat{k}_{n-2} r_{n-1}} u_{n-2}, \eta_{n-1}^{-2\hat{k}_{n-1} r_{n-1}} u_{n-1}, u_n), \\
\eta_n(u_1, \ldots, u_n) &= (u_1, \ldots, u_{n-3}, {\eta_n}^{\hat{k}_{n-2} r_n} u_{n-2}, u_{n-1}, \eta_n^{-2\hat{k}_n r_n} u_n).
\end{aligned}$$

Let $u = u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n} \in \mathbb{C}[M']$. Hence under the action we obtain

$$\begin{aligned}
\eta_i(u) &= \eta_i^{(a_{i-1}\hat{k}_{i-1}r_i - 2a_i\hat{k}_i r_i + a_{i+1}\hat{k}_{i+1}r_i)} u \quad \text{for } i \text{ even and } 2 \leq i \leq n-3, \\
\eta_{n-2}(u) &= \eta_{n-2}^{(a_{n-3}\hat{k}_{n-3}r_{n-2} - 2a_{n-2}\hat{k}_{n-2}r_{n-2} + a_{n-1}\hat{k}_{n-1}r_{n-2} + a_n\hat{k}_n r_{n-2})} u, \\
\eta_{n-1}(u) &= \eta_{n-1}^{(a_{n-2}\hat{k}_{n-2}r_{n-1} - 2a_{n-1}\hat{k}_{n-1}r_{n-1})} u, \\
\eta_n(u) &= \eta_n^{(a_{n-2}\hat{k}_{n-2}r_n - 2a_n\hat{k}_n r_n)} u.
\end{aligned}$$

To determine the ring of invariants it suffices to find the smallest $a_1, \ldots, a_n$ satisfying $\eta_i(u) = u$, where $u = u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$.

(i) Assume that n is even. It is easy to see that $a_{2j+1} \equiv 0, 1 \mod 2$ for $j = 0, \ldots, \frac{n}{2} - 1$ and $a_n \equiv 0, 1 \mod 2$. Hence u_{2j} for $j = 1, \ldots, n/2 - 1$, u_{2j-1}^2 for $j = 1, \ldots, n/2$ and u_n^2 are generators of $\mathbb{C}[M']^{\frac{N}{N'}}$. We find the remaining generators by solving the following system of equations obtained by $\eta_i(u) = u$:

$$\begin{aligned}
a_{i-1} + a_{i+1} &= 2l_i \quad \text{for } 2 \leq i \leq n-4 \text{ and even}, \\
a_{n-3} + a_{n-1} + a_n &= 2l_{n-2}
\end{aligned}$$

for some l_i. Since $(0, \ldots, 0, 1, 1)$, $(1, \ldots, 1, 0, 1)$ and $(1, \ldots, 1, 1, 0)$ are solutions of the system for $(a_1, a_3, \ldots, a_{n-5}, a_{n-3}, a_{n-1}, a_n)$, $u_{n-1}u_n, vu_n, vu_{n-1}$ are generators of the ring respectively, where $v = u_1 u_3 \cdots u_{n-3}$. The generators corresponding to other solutions are produced by the ones obtained before. This gives (i).

(ii) Assume that n is odd. We follow the same process as in the case (i) above. First, we observe that $a_{2j+1} \equiv 0, 1 \mod 2$ for $j = 0, \ldots, \frac{(n-3)}{2}$ and $a_{n-1}, a_n \equiv 0, 1, 2, 3 \mod 4$. This gives that u_{2j+1}^2, $(0 \leq j \leq \frac{(n-3)}{2})$, u_{2j}, $(1 \leq j \leq \frac{(n-3)}{2})$ and u_{n-1}^4, u_n^4 are generators of $\mathbb{C}[M']^{\frac{N}{N'}}$. The remaining generators can be found by solving the following system of equations obtained by $\eta_i(u) = u$:

$$\begin{aligned}
a_{i-1} + a_{i+1} &= 2l_i \quad \text{for } i \text{ even and } 2 \leq i \leq n-3, \\
a_{n-1} + a_n &= 4l_{n-2}, \\
-a_{n-2} + a_{n-1} &= 2l_{n-1}, \\
-a_{n-2} + a_n &= 2l_n
\end{aligned}$$

for some l_i. The solutions $(0,\ldots,0,2,2)$, $(1,\ldots,1,3,1)$ and $(1,\ldots,1,1,3)$ of the system for $(a_1,a_3,\ldots,a_{n-4},a_{n-2},a_{n-1},a_n)$ give the generators $u_{n-1}^2u_n^2$, $vu_{n-1}u_n^3$, $vu_{n-1}^3u_n$ respectively, where $v=u_1u_3\cdots u_{n-4}u_{n-2}$. The generators corresponding to other solutions are produced by the solutions obtained previously. Therefore, we get (ii). $\square$

III. Let Γ be a weighted graph of type E_n for $n=6,7,8$. The incidence matrix $\mathcal{M}(E_n)$ is defined by $(C_i\cdot C_i)=-2$ for $i=1,\ldots,n$, $(C_3\cdot C_n)=1$, $(C_i\cdot C_j)=1$ for $i=1,\ldots,n-2$ if $j=i+1$ and 0 otherwise. Since M' is generated by the rows of $\mathcal{M}(E_n)$ multiplied by $\hat{k}_i=\frac{-k_i}{\det\mathcal{M}(E_n)}$ the algebra $\mathbb{C}[M']=\mathbb{C}[u_1,\ldots,u_n]$ is defined by

$$u_1=x_1^{2\hat{k}_1}x_2^{\hat{k}_1},u_2=x_1^{\hat{k}_2}x_2^{2\hat{k}_2}x_3^{\hat{k}_2},u_3=x_2^{\hat{k}_3}x_3^{2\hat{k}_3}x_4^{\hat{k}_3}x_n^{\hat{k}_3},u_4=x_3^{\hat{k}_4}x_4^{2\hat{k}_4}x_5^{\hat{k}_4},\ldots,$$
$$u_{n-2}=x_{n-3}^{\hat{k}_{n-2}}x_{n-2}^{2\hat{k}_{n-2}}x_{n-1}^{\hat{k}_{n-2}},u_{n-1}=x_{n-2}^{\hat{k}_{n-1}}x_{n-1}^{2\hat{k}_{n-1}},u_n=x_3^{\hat{k}_n}x_n^{2\hat{k}_n}.$$

Remark 3.9 *We will use the notation* $F_i'=-\hat{k}_i(a_1,\ldots,a_n)$ *to denote the generator* $F_i'=-\hat{k}_i(a_1C_1+\ldots+a_nC_n)$. *By the formula* $\mathcal{M}(E_n)\cdot F_i'=\delta_i$ *with* $F_i'=-\hat{k}_iF_i$ *for* $i=1,\ldots,n$, *we obtain:*

(i) *If* $n=6$, *then* $\det\mathcal{M}(E_6)=3$ *and*

$$F_1'=\frac{1}{3}(4,5,6,4,2,3),F_2'=\frac{1}{3}(5,10,12,8,4,6),F_3'=(2,4,6,4,2,3),$$
$$F_4'=\frac{1}{3}(4,8,12,10,5,6),F_5'=\frac{1}{3}(2,4,6,5,4,3),F_6'=(1,2,3,2,1,2).$$

This gives $(k_1,\ldots,k_6)=(1,1,3,1,1,3)$.

(ii) *If* $n=7$, *then we have* $\det\mathcal{M}(E_7)=-2$ *and*

$$F_1'=(2,3,4,3,2,1,2),F_2'=(3,6,8,6,4,2,4),F_3'=(4,8,12,9,6,3,6),$$
$$F_4'=\frac{-1}{-2}(6,12,18,15,10,5,9),F_5'=(2,4,6,5,4,2,3),$$
$$F_6'=\frac{-1}{-2}(2,4,6,5,4,3,3),F_7'=\frac{-1}{-2}(4,8,12,9,6,3,7).$$

This gives $(k_1,\ldots,k_7)=(-2,-2,-2,-1,-2,-1,-1)$.

(iii) *If* $n = 8$, *then we have* $\det \mathcal{M}(E_8) = 1$ *and*

$$F'_1 = (4, 7, 10, 8, 6, 4, 2, 5), F'_2 = (7, 14, 20, 16, 12, 8, 4, 10),$$
$$F'_3 = (10, 20, 30, 24, 18, 12, 6, 15), F'_4 = (8, 16, 24, 20, 15, 10, 5, 12),$$
$$F'_5 = (6, 12, 18, 15, 12, 8, 4, 9), F'_6 = (4, 8, 12, 10, 8, 6, 3, 6),$$
$$F'_7 = (2, 4, 6, 5, 4, 3, 2, 3), F'_8 = (5, 10, 15, 12, 9, 6, 3, 8).$$

This gives $(k_1, \ldots, k_8) = (1, \ldots, 1)$.

The finite group N/N' in these three cases is defined as the following:

Proposition 3.10 *With the preceding notation, we have:*

(i) *If* Γ *is of type* E_6, *then* N/N' *is generated by* $\overline{C}_1, \ldots, \overline{C}_5$ *over* $\mathbb{Z}$, *where* $r_i = 3$ *for* $i = 1, \ldots, 5$.

(ii) *If* Γ *is of type* E_7, *then* N/N' *is generated by* $\overline{C}_3$ *and* $\overline{C}_5$ *over* $\mathbb{Z}$, *where* $r_3 = r_5 = 2$. *That is, the quotient is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) *If* Γ *is of type* E_8, *then* $\frac{N}{N'} =< \overline{0} >$.

Proof. As in the cases of A_n and D_n (see propositions 3.4 and 3.7), we find:

$$(a_1, \ldots, a_n)^t = \frac{1}{\det \mathcal{M}(E_n)} \mathrm{Diag}(-k_1, -k_2, \ldots, -k_n) \cdot \mathcal{M}(E_n) \cdot (b_1, \ldots, b_n)^t. \tag{3.3}$$

Since the order r_i of $\overline{C}_i$ in N/N' is the smallest $b_i \in \mathbb{Z}$ satisfying the equation (3.3), we have:

(*i*) If Γ is of type E_6, then we find $b_i = 3$ if $i \neq 6$ and $b_6 = 1$.

(*ii*) If Γ is of type E_7, then $b_2 = b_5 = 2$ and the others are 1.

(*iii*) If Γ is of type E_8, then we have $b_i = 1$ for $i = 1, \ldots, 8$. $\square$

The action of the finite group N/N' on $u = u_1^{a_1} \cdots u_n^{a_n}$ (see the proof of theorems 3.6 and 3.8) is:

$$
\begin{aligned}
\eta_1(u) &= \eta_1^{-2\hat{k}_1 r_1 a_1 + \hat{k}_2 r_1 a_2} u, \\
\eta_2(u) &= \eta_2^{\hat{k}_1 r_2 a_1 - 2\hat{k}_2 r_2 a_2 + \hat{k}_3 r_2 a_3} u, \\
\eta_3(u) &= \eta_3^{\hat{k}_2 r_3 a_2 - 2\hat{k}_3 r_3 a_3 + \hat{k}_4 r_3 a_4 + \hat{k}_n r_3 a_n} u, \\
\eta_4(u) &= \eta_4^{\hat{k}_3 r_4 a_3 - 2\hat{k}_4 r_4 a_4 + \hat{k}_5 r_4 a_5} u, \\
&\vdots \\
\eta_{n-2}(u) &= \eta_{n-2}^{\hat{k}_{n-3} r_{n-2} a_{n-3} - 2\hat{k}_{n-2} r_{n-2} a_{n-2} + \hat{k}_{n-1} r_{n-2} a_{n-1}} u, \\
\eta_{n-1}(u) &= \eta_{n-1}^{\hat{k}_{n-2} r_{n-1} a_{n-2} - 2\hat{k}_{n-1} r_{n-1} a_{n-1}} u, \\
\eta_n(u) &= \eta_n^{\hat{k}_3 r_n a_3 - 2\hat{k}_n r_n a_n} u.
\end{aligned} \tag{3.4}
$$

Hence we obtain:

Theorem 3.11 *If σ is the cone defined by the semigroup of Lipman $\mathcal{E}^+(E_n)$, $(n = 6, 7, 8)$, then the algebra $\mathbb{C}[\check{\sigma} \cap M]$ is generated by:*

(i) $u_1^3, u_2^3, u_3, u_4^3, u_5^3, u_6, u_1u_2^2u_4u_5^2, u_1^2u_2u_4^2u_5$ *for $n = 6$;*

(ii) $u_1, u_2, u_3, u_5, u_4^2, u_6^2, u_7^2, u_4u_6u_7$ *for $n = 7$;*

(iii) $u_1, \ldots, u_8$ *for $n = 8$.*

Proof. (i) By the equalities (3.4) above, we deduce

$$
\eta_1(u) = \eta_1^{2a_1 - a_2} u, \eta_2(u) = \eta_2^{-a_1 + 2a_2} u, \eta_3(u) = \eta_3^{-a_2 - a_4} u,
$$
$$
\eta_4(u) = \eta_4^{2a_4 - a_5} u, \eta_5(u) = \eta_5^{-a_4 + 2a_5} u,
$$

where $u = u_1^{a_1} \cdots u_6^{a_6}$. As before, we need to find the smallest a_k satisfying $\eta_i(u) = u$ for all i in order to obtain the ring of invariants. This condition gives the following five equations:

$$
2a_1 - a_2 = 3l_1, -a_1 + 2a_2 = 3l_2, a_2 + a_4 = 3l_3, \tag{3.5}
$$
$$
2a_4 - a_5 = 3l_4 \quad \text{and} \quad -a_4 + 2a_5 = 3l_5.
$$

for some l_i. By using the action, we observe that a_1, a_2, a_4, a_5 can be $0, 1, 2$ mod 3 and $a_3, a_6 = 1$. Then, u_1^3,u_2^3,u_4^3,u_5^3,$u_3, u_6 \in \mathbb{C}[M']^{\frac{N}{N'}}$. For the second part of (i), we have to solve the system (3.5) for each value of a_1: If $a_1 = 1$, then we have $(a_1, a_2, a_4, a_5) = (1, 2, 1, 2)$ which gives $u_1 u_2^2 u_4 u_5^2 \in \mathbb{C}[M']^{\frac{N}{N'}}$. If $a_1 = 2$, then we have $(a_1, a_2, a_4, a_5) = (2, 1, 2, 1)$. This gives $u_1^2 u_2 u_4^2 u_5 \in \mathbb{C}[M']^{\frac{N}{N'}}$. The case of $a_1 = 0$ is included in the first part.

(ii) The action is defined by:

$$\begin{aligned} \eta_3(u_1, \ldots, u_7) &= (u_1, u_2, u_3, \eta_3^{-1} u_4, a_5, u_6, \eta_3^{-1} u_7), \\ \eta_5(u_1, \ldots, u_7) &= (u_1, u_2, u_3, \eta_5^{-1} u_4, u_5, \eta_5^{-1} u_6, u_7). \end{aligned}$$

This gives $\eta_3(u) = \eta_3^{-a_4 - a_7} u$ and $\eta_5(u) = \eta_5^{-a_4 - a_6} u$, where $u = u_1^{a_1} \cdots u_7^{a_7}$. The equations obtained by $\eta_j(u) = u$ for $j = 3, 5$ give

$$a_4 + a_7 = 2l_1 \quad \text{and} \quad a_4 + a_6 = 2l_2 \tag{3.6}$$

for some l_i. Hence a_4, a_6, a_7 can be $0, 1 \mod 2$ and the smallest a_1, a_2, a_3, a_5 are 1. Therefore, $u_4^2, u_6^2, u_7^2, u_1, u_2, u_3, u_5 \in \mathbb{C}[M']^{\frac{N}{N'}}$, which is the first part of (ii). The other solution satisfying (3.6) for (a_4, a_6, a_7) is only $(1, 1, 1)$, and this gives $u_4 u_6 u_7 \in \mathbb{C}[M']^{\frac{N}{N'}}$.

(iii) The action is trivial. Therefore $\mathbb{C}[M']^{\frac{N}{N'}} = \mathbb{C}[u_1, \ldots, u_8]$. So, the corresponding toric variety is smooth.

This gives the theorem. □

We complete writing out explicitly $\mathbb{C}[\check{\sigma} \cap M]$ for ADE trees. By using [13], we can find easily the corresponding toric variety $\operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ in each cases above.

Conclusion. By the construction above, we obtain a class of toric varieties. In a future work, we will try to see whether any affine toric variety having quotient singularity corresponds to a singular graph as above and, whether we can extract some invariants of the normal surface singularity from the toric variety constructed.

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