

Some surface singularities obtained via Lie algebras

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In this work, we present a relation between a special six dimensional Lie algebra and the simple elliptic singularities of type \tilde{D}_5 .

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1. Introduction

The very well known connection between simple singularities of surfaces (in the literature, also called rational double points, Du Val singularities or Kleinian singularities) and the nilpotent varieties of simple Lie algebras was conjectured by A. Grothendieck 3 and solved by E. Brieskorn 1. A very nice consequence of that connection is to construct semi-universal deformations of simple singularities by using the corresponding Lie algebras (cf. 1, 11).

To get a similar relation for the simple elliptic singularities of surfaces, defined in 8, K. Saito constructed the elliptic root system (see ?). Since then, the simple elliptic singularities of type \tilde{E}_6 , \tilde{E}_7 and \tilde{E}_8 and, their semi-universal deformations are studied by many mathematicians.

Here we establish a relation between a special six dimensional Lie algebra and the simple elliptic singularities of type \tilde{D}_5 and construct semi-universal deformation of this singularity by using the geometry of the corresponding Lie algebra.

2. Simple and simple Elliptic Singularities

A surface singularity is a germ $(S, 0)$ defined as

$$(S, 0) = (\{x \in \mathbb{C}^N \mid f_1(x) = \dots = f_k(x) = 0\}, 0)$$

where, for each i , $f_i : \mathbb{C}^N \rightarrow \mathbb{C}$ is a germ of a holomorphic function. Assume that $(S, 0)$ is a normal surface singularity (means the local ring $\mathcal{O}_{S,0}$ is normal). A resolution of $(S, 0)$ is a map $\pi : X \rightarrow (S, 0)$ such that

(i) the surface X is nonsingular,

(ii) the map π is proper,

(iii) the restriction of π to $\pi^{-1}(S - 0)$ is an isomorphism.

A resolution is called minimal if any other resolution of $(S, 0)$ factorizes via this resolution. Minimal resolution exists and unique. The fiber $\pi^{-1}(0)$ is called the exceptional divisor of π which is, by Zariski's Main theorem, connected and 1-dimensional.

The singularity of $(S, 0)$ is called *simple singularity* if and only if it is defined by one of the following equations:

$$A_n \quad x^{n+1} + y^2 + z^2 = 0, \quad (n \geq 1)$$

$$D_n \quad x^{n-1} + xy^2 + z^2 = 0, \quad (n \geq 4)$$

$$E_6 \quad x^4 + y^3 + z^2 = 0$$

$$E_7 \quad x^3y + y^3 + z^2 = 0$$

$$E_8 \quad x^5 + y^3 + z^2 = 0$$

By 2, the exceptional divisor is, in each of cases above, a union of projective lines intersecting each other transversally and the dual graph of the minimal resolution is the Dynkin diagram having the same name.

The second class of singularities of surfaces which are relatively simple among other singularities than those given above, is the *simple elliptic singularities* defined by K. Saito in 8. A singularity of $(S, 0)$ is called simple elliptic singularity if and only if it is defined by one of the following equations:

$$\tilde{E}_6 \quad x^6 + y^3 + z^2 + \lambda xyz = 0$$

$$\tilde{E}_7 \quad x^4 + y^4 + z^2 + \lambda xyz = 0$$

$$\tilde{E}_8 \quad x^3 + y^2 + \lambda xyz = 0$$

$$\tilde{D}_5 \quad x^2 + y^2 + \lambda zw = 0$$

$$xy + z^2 + w^2 = 0$$

where $\lambda \in \mathbb{C}$ such that these equations define an isolated singularity. For

each type, the exceptional divisor is a nonsingular curve with genus 1 and, it has self-intersection -1 , -2 , -3 and -4 for \tilde{E}_6 , \tilde{E}_7 , \tilde{E}_8 and \tilde{D}_5 respectively.

Let us construct semi-universal deformations of the singularities given above: A deformation of a variety (V, v) is a map $\phi : (\mathcal{X}, p) \rightarrow (T, t)$ such that ϕ is a flat morphism with $(\phi^{-1}(t), p) \cong (V, v)$. A deformation ϕ of (V, v) is called semi-universal deformation if and only if any deformation $\phi' : (\mathcal{X}', p') \rightarrow (T', t')$ of (V, v) can be induced, up to isomorphism, from ϕ by a base change $\rho : (T', t') \rightarrow (T, t)$, and the tangent map $d\rho$ is uniquely determined.

Theorem 2.1. (see 10) *For any isolated singularity, there exists a semi-universal deformation.*

Let us first construct semi-universal deformations of an isolated hypersurface singularity $(S, 0)$: The vector space

$$T^1 := \mathbb{C}\{x_1, \dots, x_n\} / \langle f, \partial f / \partial x_1, \dots, \partial f / \partial x_n \rangle$$

is finite dimensional. Let g_1, \dots, g_k be a basis over \mathbb{C} of this space. Then

$$\mathcal{X} = \{(x_1, \dots, x_n) \times (t_1, \dots, t_k) \in \mathbb{C}^n \times \mathbb{C}^k \mid f(x_1, \dots, x_n) + \sum_{i=1}^k t_i g_i(x_1, \dots, x_n)\}$$

is the semi-universal deformation of $(S, 0)$.

Now we want to construct semi-universal deformations of a singularity of type \tilde{D}_5 . More generally, we assume that $(S, 0)$ is an r -dimensional isolated complete intersection singularity defined by $n - r$ equations. Hence we consider a free module $\mathbb{C}\{x_1, \dots, x_n\}^{n-r}$ over $\mathbb{C}\{x_1, \dots, x_n\}$ with the basis $e_1 = (1, 0, \dots, 0)^t, \dots, e_{n-r} = (0, 0, \dots, 1)^t$ and, the basis over \mathbb{C} of the finite dimensional vector space

$$T^1 := \mathbb{C}\{x_1, \dots, x_n\}^{n-r} / I$$

where

$$I = \langle f_i e_j, (\partial f_1 / \partial x_1, \dots, \partial f_{n-r} / \partial x_1)^t, \dots, (\partial f_1 / \partial x_n, \dots, \partial f_{n-r} / \partial x_n)^t \rangle$$

with $1 \leq i, j \leq n - r$. It can be easily proved that

Proposition 2.1. (see 12) *For each \tilde{D}_5 -singularity, $\dim T^1 = 7$.*

In the next sections, we will obtain semi-universal deformations of the simple and simple elliptic singularities by using the corresponding Lie algebras.

3. Semi-universal deformations via Lie algebras

The reader can find the details on the results of this section in the excellent book of P. Slodowy (11).

Let G be a semisimple complex Lie group and \mathfrak{g} be the corresponding Lie algebra over \mathbb{C} , i.e. $\mathfrak{g} = T_e G$ with $e \in G$ neutral element. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} of dimension r . On \mathfrak{h} , we have a natural action of the Weil group W . We have the following result due to Chevalley:

Theorem 3.1. (cf. 11) *The homomorphism of restriction $\mathbb{C}[\mathfrak{g}] \rightarrow \mathbb{C}[\mathfrak{h}]$ induces an isomorphism of the algebras $\mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[\mathfrak{h}]^W$.*

Hence we have the embedding $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow \mathbb{C}[\mathfrak{g}]$, so a natural morphism

$$\gamma : \mathfrak{g} \rightarrow \mathfrak{h}/W$$

This map is called the adjoint quotient map. Moreover, $\mathbb{C}[\mathfrak{g}]^G$ is generated by r homogeneous G -invariant polynomials $\gamma_1, \dots, \gamma_r$. Then we obtain a polynomial map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathbb{C}^r \\ x &\mapsto (\gamma_1(x), \dots, \gamma_r(x)) \end{aligned}$$

Theorem 3.2. (see 4,11) *With the preceding notation, each fiber of γ*
(i) consists of finitely many orbits.
(ii) has codimension r in \mathfrak{g} .
(iii) contains a unique regular orbit which is dense in the fibre.

Note that an element of \mathfrak{g} is called regular if its orbit dimension is maximal among all G -orbits. This is equivalent to say that the centralizer of a regular element in \mathfrak{g} is of minimal dimension. This gives that the regular orbits have codimension $\geq r+2$ in \mathfrak{g} . The elements in \mathfrak{g} whose G -orbits is of codimension $r+2$ are called *subregular*.

Definition 3.1. The fiber $\gamma^{-1}(\gamma(0))$ is called the nilpotent variety of \mathfrak{g} . We will denote it by $\mathcal{N}(\mathfrak{g})$.

For example, when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the nilpotent variety $\mathcal{N}(\mathfrak{g})$ is isomorphic to $\{(a, b, c) \in \mathbb{C}^3 \mid a^2 + bc = 0\}$.

Definition 3.2. A local submanifold $\mathcal{S} \subset \mathfrak{g}$ of dimension $r+2$ is called a transversal slice if \mathcal{S} intersects $\mathcal{N}(\mathfrak{g})$ transversally at the subregular element of \mathfrak{g} .

Theorem 3.3. (see 1) Let \mathfrak{g} be a Lie algebra of type A_n, D_n, E_6, E_7 or E_8 . With preceding notation,

- (i) The intersection $\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}$ is a surface singularity of the same type as \mathfrak{g} .
- (ii) The restriction of γ to \mathcal{S} is a semi-universal deformation of the singularity of $\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}$.

A detailed proof of the theorem can be found in 11.

Then the natural question arising from Brieskorn's theorem is whether $\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}$ for a finite dimensional Lie algebra \mathfrak{g} has always a nice singularity or simply, when it has a singularity. For example, when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the nilpotent variety $\mathcal{N}(\mathfrak{g})$ has an A_1 -singularity at 0 or, when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$, $\mathcal{N}(\mathfrak{g})$ is isomorphic to $\{(a, b, c) \in \mathbb{C}^3 \mid a^2 + bc = 0\} \times \mathbb{C}$, so it has a cDV -singularity at $(0, 0)$, i.e. there exists a hyperplane section having a simple singularity at $(0, 0)$ (see 7).

Here we will deal with the six dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$: The nilpotent variety $\mathcal{N} := \mathcal{N}(\mathfrak{g})$ of \mathfrak{g} is

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a^2 + bc = 0 \right\} \times \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \mid d^2 + ef = 0 \right\}.$$

The Lie group $G = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ acts on \mathcal{N} and \mathcal{N} can be decomposed into G -orbits in the following way:

$$\mathcal{N} = (\mathcal{O}_{\mathrm{reg}}^1 \times \mathcal{O}_{\mathrm{reg}}^2) \cup (\mathcal{O}_{\mathrm{reg}}^1 \times \{O\}) \cup (\{O\} \times \mathcal{O}_{\mathrm{reg}}^2) \cup \{(O, O)\}$$

For $p \in \mathcal{N}$, let us take a 4-dimensional affine subspace \mathcal{S} of \mathfrak{g} passing through p . We say that \mathcal{S} is a *generic slice at p* if its intersection with \mathcal{N} at p gives an isolated singularity (see 6 for a detailed definition of genericity). In the sequel, we denote by \mathcal{S} a generic slice at the given point. It is obvious that if $p \in \mathcal{O}_{\mathrm{reg}}^1 \times \mathcal{O}_{\mathrm{reg}}^2$, then the surface $\mathcal{N} \cap \mathcal{S}$ is non-singular at p and, if $p \in \mathcal{O}_{\mathrm{reg}}^1 \times \{O\}$ or $p \in \{O\} \times \mathcal{O}_{\mathrm{reg}}^2$, then $\mathcal{N} \cap \mathcal{S}$ has an A_1 -singularity at p .

Proposition 3.1. (see 6) With preceding notation, let $p = (0, 0)$. The surface singularity $(X, 0) := (\mathcal{N} \cap \mathcal{S}, 0)$ is a simple elliptic singularity.

Proof. Let \tilde{S} be the blowing up of $S \cong \mathbb{C}^4$ at 0. By taking the strict transform \tilde{X} of X , we have

$$\begin{array}{ccc} \mathbb{P}^3 \subset \tilde{S} & \rightarrow & S \\ \cup & & \cup \\ E \subset \tilde{X} & \rightarrow & X, \end{array}$$

where E is the exceptional curve. Since X is defined by two quadratic equations in S , the exceptional curve E will be defined by two generic quadratic equations in \mathbb{P}^3 . Hence E is an elliptic curve. Moreover, the pull-back by $\mu : \tilde{X} \rightarrow X$ of the divisor $\text{div}\varphi$ of a generic linear function φ on S can be written as $\mu^*(\text{div}\varphi) = E + C$, where C is the strict transform of $\text{div}\varphi$. Since $(\mu^*(\text{div}\varphi) \cdot E) = 0$ we obtain $E^2 = -(C \cdot E)$. By considering the intersection of E and the hyperplane associated to C in \mathbb{P}^3 , we obtain that $E \cdot C = 4$. Therefore $E^2 = -4$. \square

Now let us consider a special transversal slice $\mathcal{S}_0 := \{c = d + e, f = a + b\} \subset \mathfrak{g}$ and denote $(X_0, 0) := (\mathcal{N} \cap \mathcal{S}_0, 0)$.

Remark 3.1. It can be easily seen that the singularity $(X_0, 0)$ is of type \tilde{D}_5 .

Now we want to construct semi-universal deformations of $(X_0, 0)$: A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is defined as

$$\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} \right\}.$$

and the adjoint quotient can be regarded as

$$\chi : \begin{array}{ccc} \mathfrak{g} & \rightarrow & \mathfrak{h}/W \cong \mathbb{C}^2 \\ \left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \right) & \mapsto & (-a^2 - bc, -d^2 - ef). \end{array}$$

where W is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let us deform the adjoint quotient χ by $(\alpha, \beta) \in \mathbb{C}^2$. So, we define $f_{(\alpha, \beta)}$ as

$$f_{(\alpha, \beta)} : \begin{array}{ccc} \mathfrak{g} & \rightarrow & \mathfrak{h}/W \cong \mathbb{C}^2 \\ \left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \right) & \mapsto & (-a^2 - bc - \alpha e, -d^2 - ef - \beta b). \end{array}$$

When $(\alpha, \beta) = (0, 0)$, we have $f_{(0,0)} = \chi$.

Now let us deform the slice \mathcal{S}_0 by $(\gamma, \delta, \varepsilon) \in \mathbb{C}^3$, we define the slice $\mathcal{S}_{(\gamma, \delta, \varepsilon)}$ as

$$\mathcal{S}_{(\gamma, \delta, \varepsilon)} := \{c = d + e + \gamma, f = a + b + \delta e + \varepsilon\}.$$

For $(\gamma, \delta, \varepsilon) = (0, 0, 0)$, we have $\mathcal{S}_{(0,0,0)} = \mathcal{S}_0$.

Theorem 3.4. (see 6) With preceding notation, consider

$$S := \mathbb{C}^2 \times \mathbb{C}^3 \times \mathfrak{h}/W = \{(\alpha, \beta) \in \mathbb{C}^2\} \times \{(\gamma, \delta, \varepsilon) \in \mathbb{C}^3\} \times \{(\lambda, \mu) \in \mathfrak{h}/W\}.$$

Let \mathcal{X} be the family of surfaces on S defined as

$$\mathcal{X} := \{(X, \alpha, \beta, \gamma, \delta, \epsilon, \lambda, \mu) \in \mathfrak{g} \times S \mid f_{(\alpha, \beta)}(X) = (\lambda, \mu), X \in \mathcal{S}_{(\gamma, \delta, \epsilon)}\}.$$

Set $o := (0, 0, 0, 0, 0, 0, 0) \in S$ and $q := (O, o) \in \mathcal{X}$. Then the morphism of germs $(\mathcal{X}, q) \rightarrow (S, o)$ gives a semi-universal deformation of (X_0, p) .

Proof. Since $f_{(\alpha, \beta)}(X) = (-a^2 - bc - \alpha e, -d^2 - ef - \beta b)$ and $c = d + e + \gamma$, $f = a + b + \delta e + \epsilon$ for $X \in \mathcal{S}_{(\gamma, \delta, \epsilon)}$, the family \mathcal{X} is defined by

$$\begin{aligned} f_1 &:= a^2 + bd + be + \gamma b + \alpha e + \lambda = 0 \\ f_2 &:= d^2 + ae + be + \delta e^2 + \epsilon e + \beta b + \mu = 0. \end{aligned}$$

The coordinate ring \mathcal{O}_{X_0} of (X_0, p) is isomorphic to $\mathbb{C}\{a, b, d, e\}/(g_1, g_2)$, where $g_1 = a^2 + bd + be$ and $g_2 = d^2 + ae + be$. The \mathbb{C} -vector space T^1 of (X_0, p) is defined as $T^1 = \mathcal{O}_{X_0}^2/M$, where M is the \mathcal{O}_{X_0} -submodule of $\mathcal{O}_{X_0}^2$ generated by the 4 vectors: $(\frac{\partial g_1}{\partial a}, \frac{\partial g_2}{\partial a})$, $(\frac{\partial g_1}{\partial b}, \frac{\partial g_2}{\partial b})$, $(\frac{\partial g_1}{\partial d}, \frac{\partial g_2}{\partial d})$ and $(\frac{\partial g_1}{\partial e}, \frac{\partial g_2}{\partial e})$. Note that we have

$$(f_1, f_2) = (g_1, g_2) + \alpha(e, 0) + \beta(0, b) + \gamma(b, 0) + \delta(0, e^2) + \epsilon(0, e) + \lambda(1, 0) + \mu(0, 1).$$

We can verify that 7 vectors appeared over $(e, 0)$, $(0, b)$, $(b, 0)$, $(0, e^2)$, $(0, e)$, $(1, 0)$, and $(0, 1)$ form a basis for T^1 . Hence $(\mathcal{X}, q) \rightarrow (S, o)$ is isomorphic to a semi-universal deformation of (X_0, p) . \square

Now we want to construct semi-universal deformation spaces for a general transversal slice: For this, consider the space $\text{Aff}(\mathfrak{g}, 4)$ of all 4-dimensional affine subspaces of \mathfrak{g} . Since any 4-dimensional affine subspace of \mathfrak{g} can be described by two linear equations, $\text{Aff}(\mathfrak{g}, 4)$ is embedded in the Grassmann variety $\text{Grass}(\dim \mathfrak{g} + 1, 2) = \text{Grass}(7, 2)$. The space of all 4-dimensional linear subspaces $\text{Grass}(\mathfrak{g}, 4)$ of \mathfrak{g} is a closed subvariety of $\text{Aff}(\mathfrak{g}, 4)$.

By Proposition 3.1, $\mathcal{N} \cap \mathcal{S}$ gives us an \tilde{D}_5 -singularity for a general S in $\text{Grass}(\mathfrak{g}, 4)$. Then we obtain:

Theorem 3.5. (see 6) Let \mathcal{S} be a general element of $\text{Grass}(\mathfrak{g}, 4)$. Let \mathcal{S}_* be a “general” 3-dimensional subvariety passing \mathcal{S} of $\text{Aff}(\mathfrak{g}, 4)$. Set

$$S := \mathbb{C}^2 \times \mathcal{S}_* \times \mathfrak{h}/W = \{(\alpha, \beta) \in \mathbb{C}^2\} \times \{\mathcal{T} \in \mathcal{S}_*\} \times \{(\lambda, \mu) \in \mathfrak{h}/W \cong \mathbb{C}^2\}.$$

and

$$\mathcal{X} := \{(X, \alpha, \beta, \mathcal{T}, \lambda, \mu) \in \mathfrak{g} \times S \mid f_{(\alpha, \beta)}(X) = (\lambda, \mu), X \in \mathcal{T}\}.$$

Then the morphism of germs $(\mathcal{X}, q) \rightarrow (S, o)$ gives us a semi-universal deformation of $(\mathcal{N} \cap \mathcal{S}, p)$, where $o = (0, 0, \mathcal{S}, 0, 0)$ and $q = (0, o)$.

Proof. The condition that any 7 vectors are linearly independent in $T^1 = \mathcal{O}_{\mathcal{N} \cap \mathcal{S}}^2/M$ is open for $\mathcal{S} \in \text{Grass}(\mathfrak{g}, 4)$. Since the condition that a given family becomes a semi-universal deformation is open, we can choose a suitable 3-dimensional subvariety of $\text{Aff}(\mathfrak{g}, 4)$. \square

4. References

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